Transfinite ordinal partition relations & and coloured finite digraphs

03E02, 05C15, 05C20

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Definition

$\alpha \to (\beta, n)$ means
$\forall c : [\alpha]^2 (\exists X \in [\alpha]^\beta : c``([X]^2) = 0 \vee \exists X \in [\alpha]^n : c``([X]^2) = 1)$.

Remark

Here we are always referring to the order-type, i.e. $[\gamma]^\delta$ is the set of all subsets of γ whose order-type is δ.

Fact

For any linear order φ we have both $\varphi \not\to (\overline{\overline{\varphi}} + 1, \omega)$ and $\varphi \not\to (\omega^*, \omega)$.

Theorem (Ernst Specker, 1956)

$\omega^2 \to (\omega^2, n)$ *for all natural n.*

Theorem (Ernst Specker, 1956)

$\omega^m \not\to (\omega^m, 3)$ *for all* $m \in \omega \setminus 3$.

Theorem (Eric Charles Milner, 1973)

$\omega^\omega \to (\omega^\omega, n)$ *for all natural n.*

Theorem (Carl Darby & Jean Ann Larson)

$\omega^{\omega^2} \to (\omega^{\omega^2}, 4)$ *but* $\omega^{\omega^2} \not\to (\omega^{\omega^2}, 5)$.

Question (Handbook of Set Theory)

Does $\omega^{\omega^3} \to (\omega^{\omega^3}, 3)$*?*

Theorem (Carl Darby)

$m \to (4)^3_{2^{32}}$ *implies* $\omega^{\omega^{\omega^{\alpha+1}}} \not\to (\omega^{\omega^{\omega^{\alpha+1}}}, m)^2$.

Theorem (Darby, Schipperus & Larson)

$\beta \geqslant \gamma \geqslant 1$ *implies* $\omega^{\omega^{\beta+\gamma}} \not\to (\omega^{\omega^{\beta+\gamma}}, 5)^2$.

Theorem (Carl Darby & Rene Schipperus)

$\beta \geqslant \gamma \geqslant \delta \geqslant 1$ *implies* $\omega^{\omega^{\beta+\gamma+\delta}} \not\to (\omega^{\omega^{\beta+\gamma+\delta}}, 4)^2$.

Theorem (Rene Schipperus)

$\beta \geqslant \gamma \geqslant \delta \geqslant \varepsilon \geqslant 1$ *implies* $\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}} \not\to (\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}}, 3)^2$.

Theorem (Paul Erdős & Richard Rado, 1956)

The partition relation $\omega l \to (\omega m, n)$—with $l, m, n < \omega$—holds true if and only if every directed graph $D = \langle l, A \rangle$ contains an independent set of size m or there is a complete subtournament S of D induced by a set of n vertices such that all triples in S are transitive.

Call the m-sized independent set I_m and the transitive digraph on n vertices L_n, then this theorem may be restated as follows:

Theorem

$r(\omega m, n) = \omega r(I_m, L_n)$.

Theorem (James Earl Baumgartner, 1974)

You may replace ω by any infinite cardinal in the theorem above.

Theorem (Jean Larson, William Mitchell, 1997)

$\forall n \in \omega \setminus 2 : r(I_n, L_3) \leqslant n^2.$

Theorem (Paul Erdős, Leo Moser, 1964)

$\forall n \in \omega \setminus 3 : r(I_2, L_n) \leqslant 2^{n-1}.$

Theorem (Jean Larson, William Mitchell, 1997)

$\forall m \in \omega \setminus 3, n \in \omega \setminus 4 : r(I_2, L_n) \leqslant u(m, n)$ *with*

$$u(m,n) := \frac{1}{2}\Big(2^{n-3}\Big(4\binom{m+n-4}{n-1} + 6\binom{m+n-5}{n-2} + 9\binom{m+n-6}{n-3}\Big)\Big) + 2^{n-4} \cdot 17\binom{m+n-6}{m-2} - 1\Big)$$

Theorem (Eva Nosal, 1974)

For $n \in \omega \setminus 3$ we have $\omega^n \rightarrow (2^{n-2}, \omega^3)$ and $\omega^n \not\rightarrow (2^{n-2}+1, \omega^3)$.

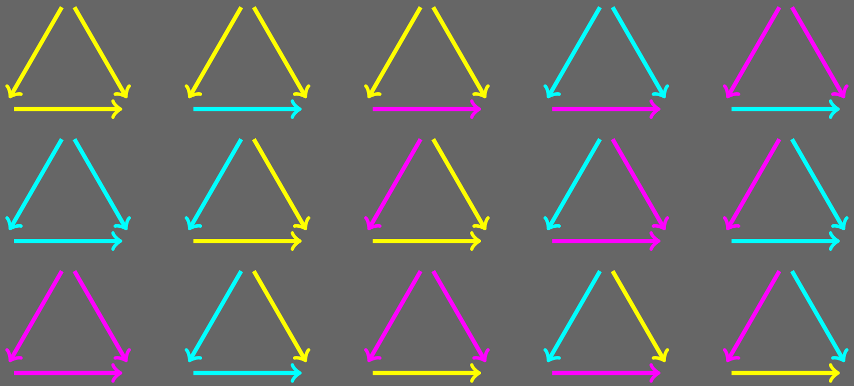
Theorem (Eva Nosal, 1979)

For $m \in \omega \setminus 5$ and $n \in \omega \setminus m$ we have $\omega^n \rightarrow (2^{\lfloor \frac{n-1}{m-1} \rfloor}, \omega^m)^2$ but $\omega^n \not\rightarrow (2^{\lfloor \frac{n-1}{m-1} \rfloor + 1}, \omega^m)^2$.

	3	4	5	6	7	8	9	m
3	6	9	14	18	23	28	36	
4	9	18	25					
ω	ω	ω	ω	ω	ω	ω	ω	ω
$\omega 2$	$\omega 4$	$\omega 8$	$\omega 14$	$\omega 28$				
$\omega 3$	$\omega 9$							
ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
$\omega^2 2$								
ω^3	ω^4	ω^4	ω^5	ω^5	ω^5	ω^5	ω^6	$\omega^{2+\lceil \mathrm{ld}(m) \rceil}$
ω^4	ω^7	ω^7	ω^{10}	ω^{10}	ω^{10}	ω^{10}		
ω^{5+n}	ω^{9+2n}	ω^{9+2n}	ω^{13+3n}	ω^{13+3n}	ω^{13+3n}	ω^{13+3n}	ω^{17+4n}	$\omega^{1+(4+n)\lceil \mathrm{ld}(m) \rceil}$
ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω
ω^{ω^2}	ω^{ω^2}	ω^{ω^2}						
$\kappa\omega 2$								
$\kappa\omega 3$								

Definition

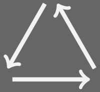
A triple is called *agreeable* if and only if it is one of the following.



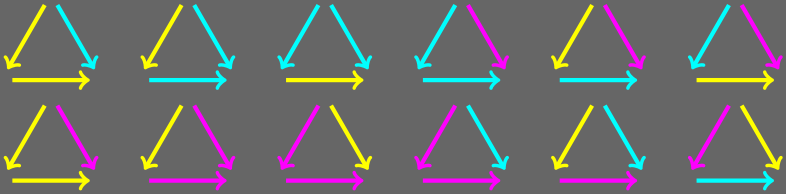
Fact

A triple is disagreeable if and only if it is either...

- *...a cyclic triple, regardless of the colouring, i.e.*



- *...or one of the following transitive triples:*



Theorem (W., 2011)

The partition relation $\omega^2 l \to (\omega^2 m, n)^2$ *holds true if and only if every edge-coloured digraph* $C = \langle I, A, c \rangle$ *with* $\operatorname{ran}(c) = 3$ *contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are agreeable.*

Call a coloured tournament on n vertices all triples of which are agreeable an A_n, then this theorem may be restated as follows:

Theorem

$r(\omega^2 m, n) = \omega^2 r(I_m, A_n).$

Does this generalize as before? Not quite, because. . .

Theorem (Erdős, Hajnal, 1971)

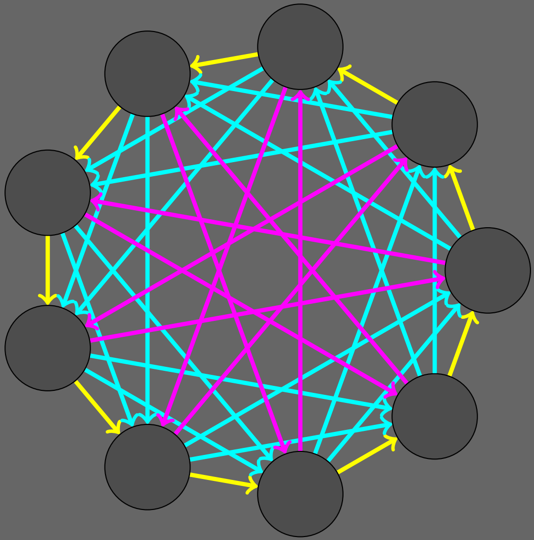
$2^\kappa = \kappa^+$ implies that $\kappa^{+2} \not\to (\kappa^{+2}, 3)$.

But in fact the above theorem also holds true for a weakly compact cardinal instead of ω, i.e.

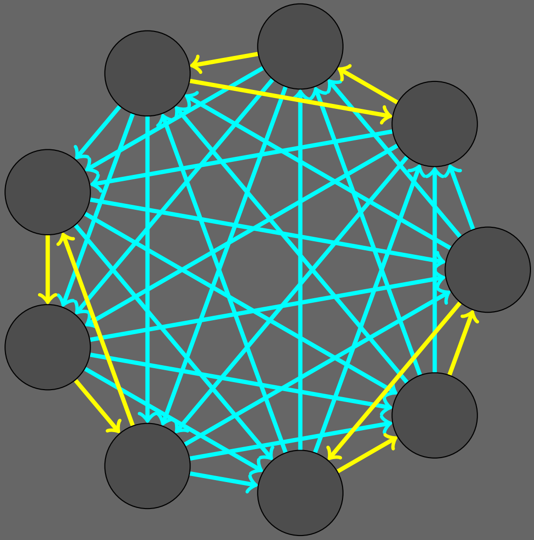
Theorem (W., 2011)

Let κ be weakly compact. Then $r(\kappa^2 m, n) = \kappa^2 r(I_m, A_n)$.

An analogue theorem and two counterexamples



An analogue theorem and two counterexamples



Proposition

- $\omega^2 r(3,3,3,3,3,3) \to (\omega^2 2, 3)$
- $r(3,3,3,3,3,3) \leqslant 1898$
- $\omega^2 r(4,4,4) \to (\omega^2 2, 3)$
- $r(4,4,4) \leqslant 236$
- $\omega^2 r(4,6) \to (\omega^2 2, 3)$
- $r(4,6) \leqslant 41$

Theorem (W., 2011)

$\omega^2 10 \to (\omega^2 2, 3)$.

Proof idea: Consider the number of arrows of each colour!

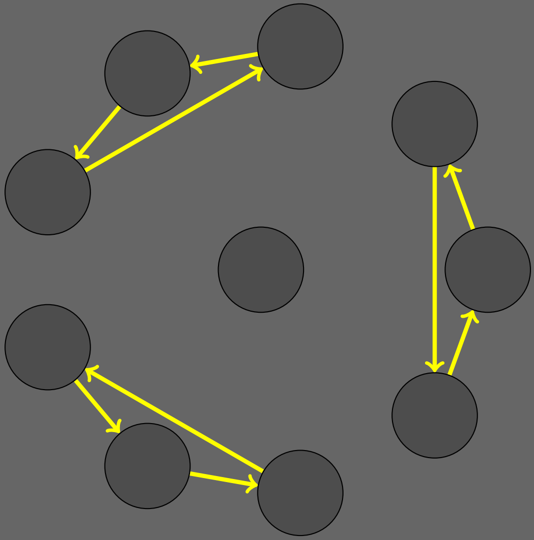
Fact

For any counterexample to $\omega^2 10 \to (\omega^2 2, 3)$:

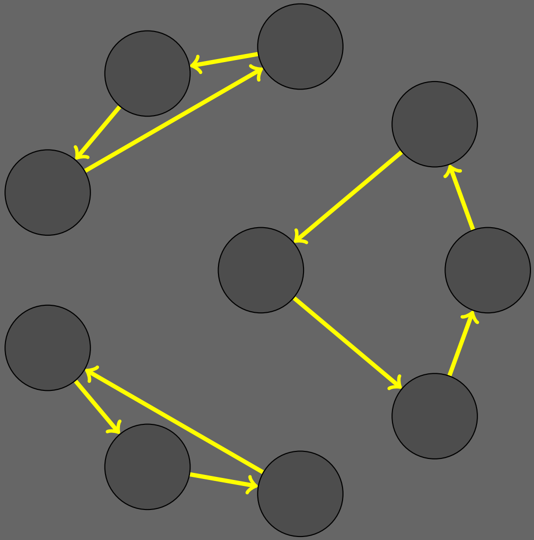
- $\# \longmapsto_t \in 11 \setminus 5.$
- $\# \longmapsto_p \in 31 \setminus 25.$
- $\# \longmapsto_y \in 11 \setminus 5.$

After this insight it is possible to reduce the structure of the turquoise arrows to two cases:

A new Ramsey number and a proof sketch



A new Ramsey number and a proof sketch



Theorem (W., 2011)

$$\forall n \in \omega \setminus 2 : r(I_n, A_3) \leqslant \frac{(2n+1)(n^2+4n-6)}{3}.$$

Corollary

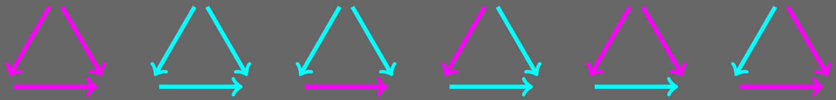
$$\forall n \in \omega \setminus 2 : r(\omega^2 n, 3) \leqslant \omega^2 \frac{(2n+1)(n^2+4n-6)}{3}.$$

Remark

We have $r(n,3), r(I_n, L_3) \in \mathcal{O}(n^2)$.

Definition

A triple is called *strongly agreeable* if and only if it is agreeable and does not contain any yellow arrow. So it is strongly agreeable precisely if it is one of these:



Theorem (W., 2011)

Let κ be weakly compact. The partition relation $\kappa\omega l \to (\kappa\omega m, n)$ holds true if and only if every coloured digraph $C = \langle l, A, c\rangle$ with $\operatorname{ran}(c) = 2$ contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are strongly agreeable.

Call a coloured tournament on n vertices all triples of which are strongly agreeable an S_n, then the theorem above may be restated as follows:

Theorem

Let κ be weakly compact. Then $r(\kappa\omega m, n) = \kappa\omega r(I_m, S_n)$.

The same works for two weakly compact cardinals of different size, i.e.

Theorem

Let κ be weakly compact and let $\lambda < \kappa$ be weakly compact. Then $r(\kappa\lambda m, n) = \kappa\lambda r(I_m, S_n)$.

Theorem (W., 2012)

For all $m \in \omega \setminus 3$ we have $r(I_m, S_3) \leqslant m(2m-1)$.

Corollary

For κ weakly compact and all $m \in \omega \setminus 3$ we have $r(\kappa\omega m, 3) \leqslant \kappa\omega m(2m-1)$.

Theorem (W., 2012)

$$\text{For any } n \in \omega \setminus 3 \text{ we have } r(I_2, S_n) \leqslant \frac{4^{n-1}+2}{3}.$$

Corollary

For κ weakly compact and any $n \in \omega \setminus 3$ we have

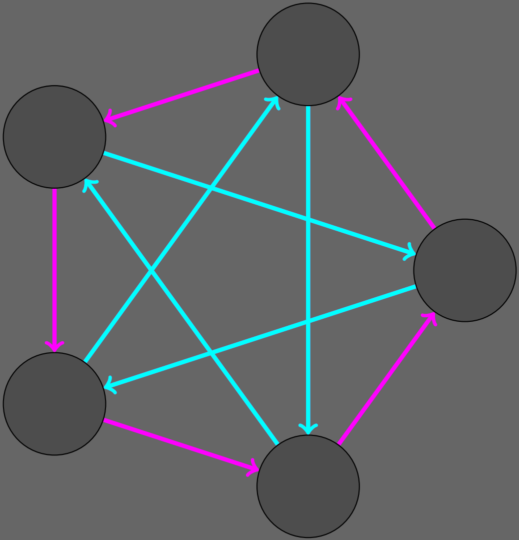
$$r(\kappa\omega 2, n) \leqslant \kappa\omega\frac{4^{n-1}+2}{3}.$$

Theorem (W., 2012)

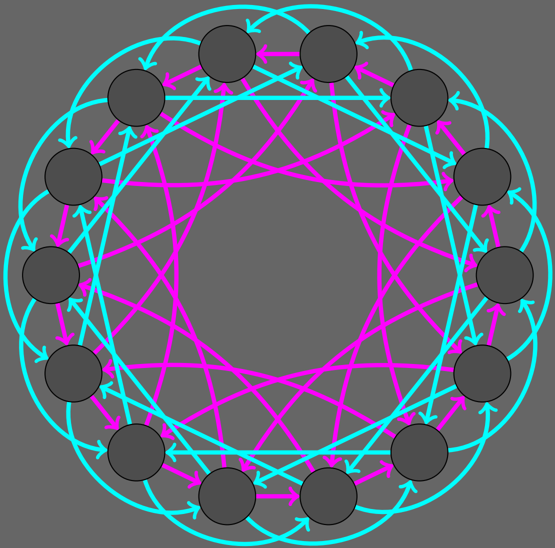
For all $m \in \omega \setminus 2$ and all $n \in \omega \setminus 3$ we have $r(I_m, S_n) \leqslant u(m,n)$ and for all weakly compact κ we have $r(\kappa\omega m, n) \leqslant \kappa\omega u(m,n)$ where

$$\begin{aligned} u(m,n) :=& \frac{1}{4}\Bigg(3 + 5 \cdot 4^{m-2}\binom{m+n-5}{m-2} \\ &- \sum_{i=3}^{m}(24i^2 - 76i + 39)4^{m-i}\binom{m+n-3-i}{m-i} \\ &+ 4^{m-1}\sum_{i=3}^{n} 4^{i-2}\binom{m+n-2-i}{n-i}\Bigg) \end{aligned}$$

Two other counterexamples



Two other counterexamples



Now we know more

	3	4	5	6	7	8	9	m
3	6	9	14	18	23	28	36	
4	9	18	25					
ω	ω	ω	ω	ω	ω	ω	ω	ω
$\omega 2$	$\omega 4$	$\omega 8$	$\omega 14$	$\omega 28$				
$\omega 3$	$\omega 9$							
ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
$\omega^2 2$	$\omega^2 10$							
ω^3	ω^4	ω^4	ω^5	ω^5	ω^5	ω^5	ω^6	$\omega^{2+\lceil \mathrm{ld}(m) \rceil}$
ω^4	ω^7	ω^7	ω^{10}	ω^{10}	ω^{10}	ω^{10}		
ω^{5+n}	ω^{9+2n}	ω^{9+2n}	ω^{13+3n}	ω^{13+3n}	ω^{13+3n}	ω^{13+3n}	ω^{17+4n}	$\omega^{1+(4+n)\lceil \mathrm{ld}(m) \rceil}$
ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω
ω^{ω^2}	ω^{ω^2}	ω^{ω^2}						
$\kappa\omega 2$	$\kappa\omega 6$							
$\kappa\omega 3$	$\kappa\omega 15$							

Question

What are $\mathcal{O}(r(I_n, A_3))$ and $\Omega(r(I_n, A_3))$?

Remark

Proving lower bounds is often difficult.

Example (Jeong Han Kim, 1995)

$r(n, 3) \in \Theta(\frac{n^2}{\log n})$

Example (Noga Alon & Vojtěch Rödl, 2005)

$r(n, 3, 3) \in \Theta(\frac{n^3}{\text{polylog } n})$.

Thank you very much for your attention!