Outline Introduction

Results by other people

Now in colour

An upper bound

Another context

Finale

Transfinite ordinal partition relations & and coloured finite digraphs 03E02, 05C15, 05C20

Thilo Weinert Hausdorff Research Centre for Mathematics, Bonn, Germany

40th Winterschool on Abstract Analysis, Section Set Theory & Topology, 2012, Sunday, January 29th, 16:00-16:40

Outline •	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale		
Table of C	ontents							
1	Introduction The con							
2	$\begin{array}{l} \alpha = \beta \\ \alpha > \beta \\ \text{Ramsey numbers} \end{array}$							
3	An ecleo An analo	lour ctical definition ogue theorem and Ramsey number and						
4 5	• Yet agai							
6	Finale Now we Open qu	know more Jestions						

Thilo Weinert, Hausdorff Research Institute for Mathematics

o ● 000000 0000000 0 0000000 000 The context	Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale
The context		•					
	The conte	xt					

Definition $\alpha \to (\beta, n)$ means $\forall c : [\alpha]^2 (\exists X \in [\alpha]^{\beta} : c^{``}([X]^2) = 0 \lor \exists X \in [\alpha]^n : c^{``}([X]^2) = 1).$

Remark

Here we are always referring to the order-type, i.e. $[\gamma]^{\delta}$ is the set of all subsets of γ whose order-type is δ .

Fact

For any linear order φ we have both $\varphi \not\rightarrow (\overline{\varphi} + 1, \omega)$ and $\varphi \not\rightarrow (\omega^*, \omega)$.

Outline Introduction o o

 $\alpha = \beta$

Results by other people

Now in colour

An upper bound

Another context

Finale

Theorem (Ernst Specker, 1956) $\omega^2 \rightarrow (\omega^2, n)$ for all natural n.

Theorem (Ernst Specker, 1956) $\omega^m \not\rightarrow (\omega^m, 3)$ for all $m \in \omega \setminus 3$.

Theorem (Eric Charles Milner, 1973) $\omega^{\omega} \rightarrow (\omega^{\omega}, n)$ for all natural n.

Theorem (Carl Darby & Jean Ann Larson) $\omega^{\omega^2} \rightarrow (\omega^{\omega^2}, 4) \text{ but } \omega^{\omega^2} \not\rightarrow (\omega^{\omega^2}, 5).$

Question (Handbook of Set Theory) Does $\omega^{\omega^3} \rightarrow (\omega^{\omega^3}, 3)$?
 Outline
 Introduction
 Results by other people
 Now in colour
 An upper bound
 Another context
 Finale

 \circ \circ

Theorem (Carl Darby) $m \to (4)^3_{2^{32}} \text{ implies } \omega^{\omega^{\omega^{\alpha+1}}} \not\to (\omega^{\omega^{\omega^{\alpha+1}}}, m)^2.$

Theorem (Darby, Schipperus & Larson) $\beta \ge \gamma \ge 1$ implies $\omega^{\omega^{\beta+\gamma}} \nrightarrow (\omega^{\omega^{\beta+\gamma}}, 5)^2$.

Theorem (Carl Darby & Rene Schipperus) $\beta \ge \gamma \ge \delta \ge 1$ implies $\omega^{\omega^{\beta+\gamma+\delta}} \nrightarrow (\omega^{\omega^{\beta+\gamma+\delta}}, 4)^2$.

Theorem (Rene Schipperus) $\beta \ge \gamma \ge \delta \ge \varepsilon \ge 1$ implies $\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}} \not\to (\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}}, 3)^2$.

Outline Introduction Results by other people Now in colour 00000

An upper bound

Finale

Theorem (Paul Erdős & Richard Rado, 1956)

The partition relation $\omega I \rightarrow (\omega m, n)$ —with I, m, n < ω —holds true if and only if every directed graph $D = \langle I, A \rangle$ contains an independent set of size m or there is a complete subtournament S of D induced by a set of n vertices such that all triples in S are transitive.

Call the *m*-sized independent set I_m and the transitive digraph on *n* vertices L_n , then this theorem may be restated as follows:

Theorem

 $\alpha > \beta$

 $r(\omega m, n) = \omega r(I_m, L_n).$

Theorem (James Earl Baumgartner, 1974) You may replace ω by any infinite cardinal in the theorem above.

Thilo Weinert, Hausdorff Research Institute for Mathematics

Transfinite ordinal partition relations & coloured digraphs

OutlineIntroductionResults by other peopleNow in colourAn upper boundAnother contextFinale \circ \circ </td

Theorem (Jean Larson, William Mitchell, 1997) $\forall n \in \omega \setminus 2 : r(I_n, L_3) \leq n^2.$

Theorem (Paul Erdős, Leo Moser, 1964) $\forall n \in \omega \setminus 3 : r(l_2, L_n) \leq 2^{n-1}.$

Theorem (Jean Larson, William Mitchell, 1997) $\forall m \in \omega \setminus 3, n \in \omega \setminus 4 : r(I_2, L_n) \leq u(m, n)$ with

$$u(m,n) := \frac{1}{2} \left(2^{n-3} \left(4 \binom{m+n-4}{n-1} + 6 \binom{m+n-5}{n-2} \right) \right. \\ \left. + 9 \binom{m+n-6}{n-3} \right) + 2^{n-4} \cdot 17 \binom{m+n-6}{m-2} - 1 \right)$$

Thilo Weinert, Hausdorff Research Institute for Mathematics

Transfinite ordinal partition relations & coloured digraphs

Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale
		000000				
lpha > eta						

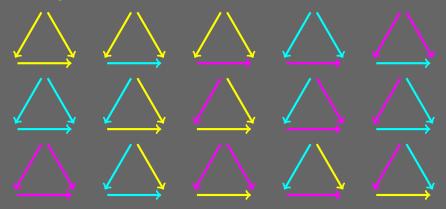
Theorem (Eva Nosal, 1974) For $n \in \omega \setminus 3$ we have $\omega^n \to (2^{n-2}, \omega^3)$ and $\omega^n \not\to (2^{n-2} + 1, \omega^3)$.

Theorem (Eva Nosal, 1979) For $m \in \omega \setminus 5$ and $n \in \omega \setminus m$ we have $\omega^n \to (2^{\lfloor \frac{n-1}{m-1} \rfloor}, \omega^m)^2$ but $\omega^n \not\to (2^{\lfloor \frac{n-1}{m-1} \rfloor+1}, \omega^m)^2$.

Outline O		uction	Results by ○○○○○●	other peopl		n colour 000000	An upper b ○		Another context	Finale
Ramsey	numbers									
		3	4	5	6	7	8	9	<i>m</i>	
	3	6	9	14	18	23	28	36		
	4	9	18	25						
	ω	ω	ω	ω	ω	ω	ω		ω	
	ω2	$\omega 4$	$\omega 8$	$\omega 14$	$\omega 28$					
	ω3	$\omega 9$								
	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	
	$\omega^2 2$									
	ω^3	ω^4	ω^4	ω^5	ω^5	ω^5	ω^5	ω^6	$\omega^{2+\lceil \operatorname{Id}(m) \rceil}$	
	ω^4	ω^7	ω^7	ω^{10}	ω^{10}	ω^{10}	ω^{10}			
	ω^{5+n}	ω^{9+2n}	ω^{9+2n}	ω^{13+3n}	ω^{13+3n}	ω^{13+3n}	ω^{13+3n}	ω^{17+4r}	$\omega^{1+(4+n)\lceil \operatorname{Id}($	<i>m</i>)]
	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	
	ω^{ω^2}	ω^{ω^2}	ω^{ω^2}							
	$\kappa\omega 2$									
	$\kappa\omega 3$									

Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale
⊖ An eclecti	o cal definition	000000	000000000	0	000000	000

Definition A triple is called *agreeable* if and only if it is one of the following.



Thilo Weinert, Hausdorff Research Institute for Mathematics

Transfinite ordinal partition relations & coloured digraphs

Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale
○ An eclectic	o al definition	000000	000000000	0	000000	000

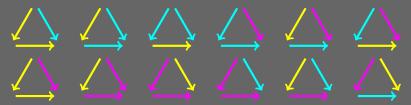
Fact

A triple is disagreeable if and only if it is either...

• ... a cyclic triple, regardless of the colouring, i.e.



• ... or one of the following transitive triples:



Outline	Introduction	Results by other people		An upper bound	Another context	Finale
ਂ An analog	ਂ ue theorem and t	oooooo wo counterexamples	00 000 0000		000000	000

Theorem (W., 2011)

The partition relation $\omega^2 I \rightarrow (\omega^2 m, n)^2$ holds true if and only if every edge-coloured digraph $C = \langle I, A, c \rangle$ with ran(c) = 3contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are agreeable.

Call a coloured tournament on n vertices all triples of which are agreeable an A_n , then this theorem may be restated as follows:

Theorem

 $r(\omega^2 m, n) = \omega^2 r(I_m, A_n).$

Does this generalize as before? Not quite, because...

Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale			
			000000000000						
An analogue theorem and two counterexamples									

Theorem (Erdős, Hajnal, 1971) $2^{\kappa} = \kappa^{+}$ implies that $\kappa^{+2} \neq (\kappa^{+2}, 3)$.

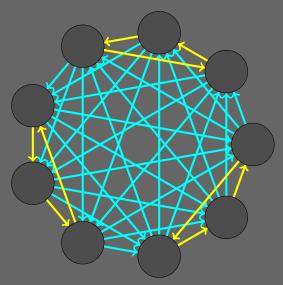
But in fact the above theorem also holds true for a weakly compact cardinal instead of ω , i.e.

Theorem (W., 2011)

Let κ be weakly compact. Then $r(\kappa^2 m, n) = \kappa^2 r(I_m, A_n)$.

Outline O	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale
		wo counterexamples				

Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale				
			0000000000							
An analogue theorem and two counterexamples										



Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale			
			00000000000						
A new Ramsey number and a proof sketch									

Proposition

•
$$\omega^2 r(3,3,3,3,3,3) o (\omega^2 2,3)$$

- $r(3, 3, 3, 3, 3, 3) \leq 1898$
- $\omega^2 r(4,4,4) \rightarrow (\omega^2 2,3)$
- $r(4, 4, 4) \leq 236$
- $\omega^2 r(4,6) \rightarrow (\omega^2 2,3)$
- $r(4,6) \leqslant 41$

Theorem (W., 2011) $\omega^2 10 \rightarrow (\omega^2 2, 3).$

Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale
⊖ A new Ra	o nsey number and	oooooo a proof sketch	000000000000		000000	000
						/

Proof idea: Consider the number of arrows of each colour!

Fact

For any counterexample to $\omega^2 10
ightarrow (\omega^2 2,3)$:

•
$$\# \mapsto_t \in 11 \setminus 5.$$

•
$$\# \mapsto_{p} \in 31 \setminus 25$$

• $\# \mapsto_{y} \in 11 \setminus 5.$

After this insight it is possible to reduce the structure of the turquoise arrows to two cases:

Outline O	Introduction O	Results by other people	Now in colour ○○○○○○○○●○	An upper bound	Another context	Finale
A new Rai	nsey number and	l a proof sketch				
				1		
			\frown			
		\frown				
		()				
		()_				

Outline O	Introduction O	Results by other people	Now in colour ○○○○○○○○○●	An upper bound O	Another context	Finale
A new Ra	msey number and	l a proof sketch				
			\frown			
				\frown		
		()				
		\bigcirc		\uparrow	ĺ	
				()		

Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale
				•		
An upper	bound					

Theorem (W., 2011)

$$\forall n \in \omega \setminus 2 : r(I_n, A_3) \leqslant \frac{(2n+1)(n^2+4n-6)}{3}$$

Corollary

$$\forall n \in \omega \setminus 2 : r(\omega^2 n, 3) \leqslant \omega^2 \frac{(2n+1)(n^2+4n-6)}{3}$$

Remark We have $r(n, 3), r(I_n, L_3) \in \mathcal{O}(n^2)$.

Outline O	Introduction O	Results by other people	Now in colour	An upper bound ○	Another context ●○○○○○	Finale
Yet anoth	er definition					
	efinition					

A triple is called *strongly agreeable* if and only if it is agreeable and does not contain any yellow arrow. So it is strongly agreeable precisely if it is one of these:

Theorem (W., 2011)

Let κ be weakly compact. The partition relation $\kappa\omega l \rightarrow (\kappa\omega m, n)$ holds true if and only if every coloured digraph $C = \langle l, A, c \rangle$ with $\operatorname{ran}(c) = 2$ contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are strongly agreeable.

Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale		
					00000			
Yet again some upper bounds								

Call a coloured tournament on n vertices all triples of which are strongly agreeable an S_n , then the theorem above may be restated as follows:

Theorem

Let κ be weakly compact. Then $r(\kappa \omega m, n) = \kappa \omega r(I_m, S_n)$.

The same works for two weakly compact cardinals of different size, i.e.

Theorem

Let κ be weakly compact and let $\lambda < \kappa$ be weakly compact. Then $r(\kappa \lambda m, n) = \kappa \lambda r(I_m, S_n)$.

Theorem (W., 2012) For all $m \in \omega \setminus 3$ we have $r(I_m, S_3) \leq m(2m-1)$.

0 0	utline	Introduction O	Results by other people	Now in colour	An upper bound O	Another context ○○●○○○	Finale		
Y	et again s	ome upper bound	ls						
	С	orollary							
	For κ weakly compact and all $m \in \omega \setminus 3$ we have $r(\kappa \omega m,3) \leqslant \kappa \omega m(2m-1).$								
	TI	heorem ((W., 2012)						
		For a	any n $\in \omega \setminus 3$ we	e have r(l ₂ ,	$S_n)\leqslant rac{4^{n-1}}{3}$	+2.			

Corollary For κ weakly compact and any $n \in \omega \setminus 3$ we have

$$r(\kappa\omega 2,n)\leqslant\kappa\omegarac{4^{n-1}+2}{3}.$$

Transfinite ordinal partition relations & coloured digraphs

Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale			
					000000				
Yet again	Yet again some upper bounds								

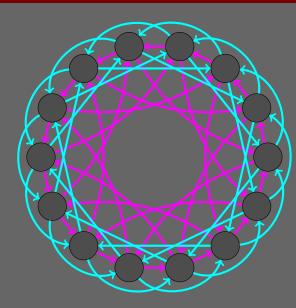
Theorem (W., 2012)

For all $m \in \omega \setminus 2$ and all $n \in \omega \setminus 3$ we have $r(I_m, S_n) \leq u(m, n)$ and for all weakly compact κ we have $r(\kappa \omega m, n) \leq \kappa \omega u(m, n)$ where

$$u(m,n) := \frac{1}{4} \left(3 + 5 \cdot 4^{m-2} \binom{m+n-5}{m-2} \right)$$
$$- \sum_{i=3}^{m} (24i^2 - 76i + 39) 4^{m-i} \binom{m+n-3-i}{m-i}$$
$$+ 4^{m-1} \sum_{i=3}^{n} 4^{i-2} \binom{m+n-2-i}{n-i} \right)$$

Outline O	Introduction O	Results by other people	Now in colour	An upper bound O	Another context ○○○○●○	Finale
Two other	r counterexamples					

Outline	Introduction	Results by other people	Now in colour	An upper bound	Another context	Finale
					000000	
Two othe	r counterexample	s				



Outline 0	Introd O		Results by	other peopl		n colour 000000	An upper b		Another context	Finale ●○○
Now we	e know mor	e								
		3	4	5	6	7	8	9	<i>m</i>	
	3	6	9	14	18	23	28	36		
	4	9	18	25						
	ω	ω	ω	ω	ω	ω	ω	ω	ω	
	$\omega 2$	ω 4	ω 8	$\omega 14$	$\omega 28$					
	ω3	$\omega 9$								
	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	
	$\omega^2 2$	$\omega^2 10$								
	ω^3	ω^4	ω^4	ω^5	ω^5	ω^5	ω^5	ω^6	$\omega^{2+\lceil \operatorname{Id}(m) \rceil}$	
	ω^4	ω^7	ω^7	ω^{10}	ω^{10}	ω^{10}	ω^{10}			
	ω^{5+n}	ω^{9+2n}	ω^{9+2n}	ω^{13+3n}		ω^{13+3n}	ω^{13+3n}	ω^{17+4r}	$\omega^{1+(4+n)\lceil \operatorname{Id}(n) \rceil}$	m)⊺
	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	ω^{ω}	
	ω^{ω^2}	ω^{ω^2}	ω^{ω^2}							
	$\kappa\omega 2$	$\kappa\omega 6$								
	$\kappa\omega 3$	$\kappa\omega 15$								

Outline O	Introduction O	Results by other people	Now in colour	An upper bound ○	Another context	Finale ○●○
Open que	stions					

Question What are $\mathcal{O}(r(I_n, A_3))$ and $\Omega(r(I_n, A_3))$?

Remark Proving lower bounds is often difficult.

Example (Jeong Han Kim, 1995) $r(n,3) \in \Theta(\frac{n^2}{\log n})$

Example (Noga Alon & Vojtěch Rödl, 2005) $r(n,3,3) \in \Theta(\frac{n^3}{\text{polylog }n}).$

Outline O	Introduction O	Results by other people	Now in colour	An upper bound ○	Another context	Finale ○○●
Gratitude						

Thank you very much for your attention!